

Maximum scattered \mathbb{F}_q -linear sets of $\text{PG}(1, q^4)$

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Abstract

There are two known families of maximum scattered \mathbb{F}_q -linear sets in $\text{PG}(1, q^t)$: the linear sets of pseudoregulus type and for $t \geq 4$ the scattered linear sets found by Lunardon and Polverino. For $t = 4$ we show that these are the only maximum scattered \mathbb{F}_q -linear sets and we describe the orbits of these linear sets under the groups $\text{PGL}(2, q^4)$ and $\text{PTL}(2, q^4)$.

1 Introduction

Recent investigations on linear sets in a finite projective line $\text{PG}(1, q^t)$ of rank t concerned: the hypersurface obtained from the linear sets of pseudoregulus type by applying field reduction [12]; a geometric characterization of the linear sets of pseudoregulus type [9]; a characterization of the clubs, that is, the linear sets of rank r with a point of weight $r - 1$ [13]; a generalization of clubs in order to construct KM-arcs [10]; a condition for the equivalence of two linear sets [8, 18]; the definition and study of the class of a linear set in order to study their equivalence [7]; a construction method which yields MRD-codes from maximum scattered linear sets of $\text{PG}(1, q^t)$ [17]. Furthermore, the linear sets in $\text{PG}(1, q^t)$ coincide with the so-called splashes of subgeometries [13]. The results of such investigations make it reasonable to attempt to classify the linear sets in $\text{PG}(1, q^t)$ of rank t for small t .

A point in $\text{PG}(1, q^t)$ is the \mathbb{F}_{q^t} -span $\langle \mathbf{v} \rangle_{\mathbb{F}_{q^t}}$ of a nonzero vector \mathbf{v} in a two-dimensional vector space, say W , over \mathbb{F}_q . If U is a subspace over \mathbb{F}_q of

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W , then $L_U = \{\langle \mathbf{v} \rangle_{\mathbb{F}_{q^t}} : \mathbf{v} \in U \setminus \{\mathbf{0}\}\}$ denotes the associated \mathbb{F}_q -linear set (or simply *linear set*) in $\text{PG}(1, q^t)$. The *rank* of such a linear set is $r = \dim_{\mathbb{F}_q} U$. Any linear set in $\text{PG}(1, q^t)$ of rank greater than t coincides with the whole projective line. The *weight* of a point $P = \langle \mathbf{v} \rangle_{\mathbb{F}_{q^t}}$ is $w(P) = \dim_{\mathbb{F}_q}(U \cap P)$. If the rank and the size of L_U are r and $(q^r - 1)/(q - 1)$, respectively, then L_U is *scattered*. Equivalently, L_U is scattered if and only if all its points have weight one. A scattered \mathbb{F}_q -linear set of rank t in $\text{PG}(1, q^t)$ is *maximum scattered*. An example of maximum scattered \mathbb{F}_q -linear set in $\text{PG}(1, q^t)$ is L_V with $V = \{(u, u^q) : u \in \mathbb{F}_{q^t}\}$. Any subset of $\text{PG}(1, q^t)$ projectively equivalent to this L_V is called *linear set of pseudoregulus type*. See [9] for a geometric description, and [7] or the survey [16] for further background on linear sets. Note that for any $\varphi \in \Gamma\text{L}(2, q^t)$ with related collineation $\tilde{\varphi} \in \text{P}\Gamma\text{L}(2, q^t)$ and any \mathbb{F}_q -linear set L_U , $L_{U^\varphi} = (L_U)^{\tilde{\varphi}}$. In [7, Theorem 4.5] it is proved that if $t = 4$ and L_U has *maximum field of linearity* \mathbb{F}_q , that is, L_U is not an \mathbb{F}_{q^s} -linear set for $s > 1$, then any linear set in the same orbit of L_U under the action of $\text{P}\Gamma\text{L}(2, q^4)$ is of type L_{U^φ} with $\varphi \in \Gamma\text{L}(2, q^4)$. Note that this is not true if $t > 4$. In [14], Lunardon and Polverino construct a class of maximum scattered linear sets:

Theorem 1.1 ([14]). *Let q be a prime power, $t \geq 4$ an integer, $b \in \mathbb{F}_{q^t}$ such that the norm $N_{q^t/q}(b)$ of b over \mathbb{F}_q is distinct from one, and*

$$U(b, t) = \{(u, bu^q + u^{q^{t-1}}) : u \in \mathbb{F}_{q^t}\}. \quad (1)$$

If $b \neq 0$ then $L_{U(b, t)}$ is a maximum scattered \mathbb{F}_q -linear set in $\text{PG}(1, q^t)$ and if $q > 3$, then it is not of pseudoregulus type.

It can be directly seen that $L_{U(0, t)}$ is maximum scattered of pseudoregulus type. For $t = 4$, Theorem 1.1 can be extended to $q = 3$, as it can be checked by using the package `FinInG` of `GAP` [3]. In the following $t = 4$ is assumed. For all $b \in \mathbb{F}_{q^4}$ define

$$U(b) = U(b, 4) = \{(x, bx^q + x^{q^3}) : x \in \mathbb{F}_{q^4}\}. \quad (2)$$

In section 2 it is shown that $N_{q^4/q}(b) \neq 1$ is a necessary condition to obtain scattered linear sets of $\text{PG}(1, q^4)$ and the case $N_{q^4/q}(b) = 1$ is dealt with. In this case, $L_{U(b)}$ contains either one or $q + 1$ points of weight two, and the remaining points have weight one.

The main result in section 3 is that if L is a maximum scattered linear set in $\text{PG}(1, q^4)$, then L is projectively equivalent to $L_{U(b)}$ for some $b \in \mathbb{F}_{q^4}$ with $N_{q^4/q}(b) \neq 1$ (cf. Theorem 3.4).

In section 4 the orbits of the \mathbb{F}_q -linear sets of rank four in $\text{PG}(1, q^4)$ of type $L_{U(b)}$, under the actions of both $\text{PGL}(2, q^4)$ and $\text{P}\Gamma\text{L}(2, q^4)$, are completely characterized. Such orbits only depend on the norm b^{q^2+1} of b over \mathbb{F}_{q^2} . In particular, $\text{PG}(1, q^4)$ contains precisely $q(q-1)/2$ maximum scattered linear sets up to projective equivalence (Theorem 4.5), one of them is of pseudoregulus type, the others are as in Theorem 1.1.

2 Classification

This section is devoted to the classification of all $L_{U(b)}$ for $b \in \mathbb{F}_{q^4}$, where $U(b)$ is as in (2).

Theorem 2.1. *For $b \in \mathbb{F}_{q^4}$ the following holds.*

1. *If $N_{q^4/q}(b) \neq 1$, then $L_{U(b)}$ is scattered.*
2. *If $N_{q^4/q^2}(b) = 1$, then $L_{U(b)}$ has a unique point with weight two, the point $\langle(1, 0)\rangle_{\mathbb{F}_{q^4}}$, and all other with weight one.*
3. *If $N_{q^4/q^2}(b) \neq 1$ and $N_{q^4/q}(b) = 1$, then $L_{U(b)}$ has $q+1$ points with weight two and all other with weight one.*

Proof. Put $f_b(x) = bx^q + x^{q^3}$. For $x \in \mathbb{F}_{q^4}^*$ the point $P_x := \langle(x, f_b(x))\rangle_{\mathbb{F}_{q^4}}$ of $L_{U(b)}$ has weight more than one if and only if there exists $y \in \mathbb{F}_{q^4}^*$ and $\lambda \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$ such that $\lambda(x, f_b(x)) = (y, f_b(y))$. This holds if and only if $y = \lambda x$ and

$$\lambda bx^q + \lambda x^{q^3} - \lambda^q bx^q - \lambda^{q^3} x^{q^3} = 0. \quad (3)$$

For a given x the solutions in λ of (3) form an \mathbb{F}_q -subspace whose rank equals to the weight of the point P_x . Since q -polynomials over \mathbb{F}_{q^4} of rank 1 are of the form $\alpha \text{Tr}_{q^4/q}(\beta x) \in \mathbb{F}_{q^4}[x]$, it is clear that the kernel of the \mathbb{F}_q -linear map in the variable λ at the left-hand side of (3) has dimension at most two and hence the weight of each point of $L_{U(b)}$ is at most two. If (λ, x) is a solution of (3) for some $\lambda \in \mathbb{F}_{q^4}$ and $x \in \mathbb{F}_{q^4}^*$, then (λ', x') is also a solution for each $\lambda' \in \langle 1, \lambda \rangle_{\mathbb{F}_q}$ and $x' \in \langle x \rangle_{\mathbb{F}_{q^2}}$ and hence for each $\mu \in \mathbb{F}_{q^2}^*$ if P_x has weight two, then $P_{\mu x} := \langle(\mu x, f_b(\mu x))\rangle_{\mathbb{F}_{q^4}}$ has weight two as well. Note that $P_{\mu x} = \langle(1, \mu^{q-1}(bx^{q-1} + x^{q^3-1}))\rangle_{\mathbb{F}_{q^4}}$ and hence if $P_x \neq \langle(1, 0)\rangle_{\mathbb{F}_{q^4}}$ has weight two, then $\{P_{\mu x} : \mu \in \mathbb{F}_{q^2}^*\}$ is a set of $q+1$ distinct points with weight 2.

The function $f_b(x)$ is not \mathbb{F}_{q^2} -linear and hence the maximum field of linearity of $L_{U(b)}$ is \mathbb{F}_q . It follows (cf. [7, Proposition 2.2]) that $L_{U(b)}$ has

at least one point with weight one, say $\langle(x_0, f_b(x_0))\rangle_{\mathbb{F}_{q^4}}$. Then the line of $\text{AG}(2, q^4)$ with equation $x_0Y = f_b(x_0)X$ meets the graph of $f_b(x)$, that is, $\{(x, f_b(x)) : x \in \mathbb{F}_{q^4}\}$, in exactly q points. It follows from [1, 2], see also [6], that the number of directions determined by $f_b(x)$ is at least $q^3 + 1$, and hence also $|L_{U(b)}| \geq q^3 + 1$. Denote by w_1 and w_2 the number of points of $L_{U(b)}$ with weight one and two, respectively. Then

$$w_1 + w_2 = |L_{U(b)}| \geq q^3 + 1, \quad (4)$$

$$w_1(q-1) + w_2(q^2-1) = q^4 - 1. \quad (5)$$

Subtracting (4) $(q-1)$ -times from (5) gives $w_2(q^2 - q) \leq q^3 - q$ and hence $w_2 \leq q + 1$. At this point it is clear that in $L_{U(b)}$ there is either one point with weight two, the point $\langle(1, 0)\rangle_{\mathbb{F}_{q^4}}$, or there are exactly $q + 1$ of them and $\langle(1, 0)\rangle_{\mathbb{F}_{q^4}}$ is not one of them.

If $N_{q^4/q}(b) \neq 1$, then Theorem 1.1 states that $L_{U(b)}$ is scattered. We show that $\langle(1, 0)\rangle_{q^4}$ has weight two if and only if $N_{q^4/q^2}(b) = 1$. Note that the weight of this point is the dimension of the kernel of $f_b(x)$. If $f_b(x) = 0$ for some $x \in \mathbb{F}_{q^4}^*$, then $b = -x^{q^3-q}$ and hence, by taking $(q^2 + 1)$ -th powers at both sides, $N_{q^4/q^2}(b) = 1$. On the other hand, if $N_{q^4/q^2}(b) = 1$, then $b = w^{q^2-1}$ for some $w \in \mathbb{F}_{q^4}^*$. Let ε be a non-zero element of \mathbb{F}_{q^4} such that $\varepsilon^{q^2} + \varepsilon = 0$. Then it is easy to check that the kernel of $f_b(x)$ is $\langle(\varepsilon w)^{q^3}\rangle_{\mathbb{F}_{q^2}}$ which has dimension two over \mathbb{F}_q and hence $\langle(1, 0)\rangle_{q^4}$ has weight two.

It remains to prove that if $N_{q^4/q}(b) = 1$ and $N_{q^4/q^2}(b) \neq 1$, then there is at least one point (hence precisely $q + 1$ points) of weight two. After rearranging in (3), we obtain

$$(\lambda - \lambda^q)^{q^3-1} = bx^{q-q^3}. \quad (6)$$

By taking $(q^2 + 1)$ -th powers on both sides we can eliminate x , obtaining

$$(\lambda - \lambda^q)^{(q^3-1)(q^2+1)} = (\lambda - \lambda^q)^{(q-1)(q^2+1)} = b^{q^2+1}. \quad (7)$$

It is clear that we can find $\lambda \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$ satisfying (7) if and only if there exists $\epsilon \in \mathbb{F}_{q^4}^*$ such that

$$(\lambda - \lambda^q)^{q^3-1}/b = \epsilon^{q^2-1}. \quad (8)$$

Then $x \in \langle\epsilon^q\rangle_{\mathbb{F}_{q^2}}$ with $y = \lambda x$ satisfies our initial conditions in (3).

Now use $N_{q^4/q}(b) = 1$ and put $b = \mu^{q-1}$ for some $\mu \in \mathbb{F}_{q^4}^*$. Then (7) can be written as

$$\left(\frac{\lambda - \lambda^q}{\mu}\right)^{(q-1)(q^2+1)} = 1. \quad (9)$$

We can solve (9) if and only if there exists $\delta \in \mathbb{F}_{q^4}^*$ such that

$$\left(\frac{\lambda - \lambda^q}{\mu}\right)^{q-1} = \delta^{q^2-1}, \quad (10)$$

or, equivalently,

$$\left\langle \frac{\lambda - \lambda^q}{\mu} \right\rangle_{\mathbb{F}_q} = \langle \delta^{q+1} \rangle_{\mathbb{F}_q}. \quad (11)$$

Now we will continue in $\text{PG}(\mathbb{F}_{q^4}, \mathbb{F}_q) = \text{PG}(3, q)$. At the left-hand side of (11) we can see a point of the hyperplane \mathcal{H}_μ defined as

$$\mathcal{H}_\mu = \{\langle z \rangle_{\mathbb{F}_q} : \text{Tr}_{q^4/q}(\mu z) = 0\},$$

while on the right-hand side we can see a point of the elliptic quadric \mathcal{Q} defined as

$$\mathcal{Q} = \{\langle z \rangle_{\mathbb{F}_q} : z^{(q-1)(q^2+1)} = 1\}.$$

For a proof that \mathcal{Q} is an elliptic quadric see [5, Theorem 3.2]. Since $\mathcal{Q} \cap \mathcal{H}_\mu \neq \emptyset$ it follows that we can always find $\lambda \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$ satisfying (8) and hence $L_{U(b)}$ is not scattered.

□

Remark 2.2. *The linear sets in Theorem 2.1 are of sizes $q^3 + q^2 + q + 1$, $q^3 + q^2 + 1$, or $q^3 + 1$. The linear set associated with $\{(x, \text{Tr}_{q^4/q}(x)) : x \in \mathbb{F}_{q^4}\}$ is of size $q^3 + 1$ as well. As it turns out from [4] the projective line $\text{PG}(1, q^4)$ also contains \mathbb{F}_q -linear sets of size $q^3 + q^2 - q + 1$.*

3 The canonical form

In this section \mathbb{L} denotes a maximum scattered \mathbb{F}_q -linear set in $\text{PG}(1, q^4)$, not of pseudoregulus type. In particular, this implies $q > 2$. By [15], \mathbb{L} is a projection $p_\ell(\Sigma)$, where the vertex ℓ is a line and Σ is a q -order canonical subgeometry¹ in $\text{PG}(3, q^4)$, with $\ell \cap \Sigma = \emptyset$. The axis of the projection

¹Let $\text{PG}(V, \mathbb{F}_{q^t}) = \text{PG}(n-1, q^t)$, let U be an n -dimensional \mathbb{F}_q -vector subspace of V , and $\Sigma = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^t}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}$. If $\langle \Sigma \rangle = \text{PG}(n-1, q^t)$, then Σ is a (q -order) *canonical subgeometry* of $\text{PG}(n-1, q^t)$. Here and in the following, angle brackets $\langle - \rangle$ without a subscript denote projective span in $\text{PG}(n-1, q^t)$, that is, $\text{PG}(3, q^4)$ in our case.

is immaterial and can be chosen by convenience. Let σ be a generator of the subgroup of order four of $\text{P}\Gamma\text{L}(4, q^4)$ fixing pointwise Σ . Let M be a k -dimensional subspace of $\text{PG}(3, q^4)$. We say that M is a *subspace of Σ* if $M \cap \Sigma$ is a k -dimensional subspace of Σ , which happens exactly when $M^\sigma = M$.

Proposition 3.1. *Let Σ' be the unique q^2 -order canonical subgeometry of $\text{PG}(3, q^4)$ containing Σ , that is, the set of all points fixed by σ^2 . Then the intersection of ℓ and Σ' is empty.*

Proof. Assume the contrary, that is, there exists a point P in $\ell \cap \Sigma'$. Then $P^{\sigma^2} = P$, the subspace $\ell_P = \langle P, P^\sigma \rangle$ is a line, and satisfies $\ell_P^\sigma = \ell_P$, whence ℓ_P is a line of Σ . This implies that $p_\ell(\ell_P)$ is a point, and \mathbb{L} is not scattered. \square

Let \mathcal{K} and \mathcal{K}' be the Klein quadrics representing – via the Plücker embedding \wp – the lines of Σ and Σ' . In order to precisely define \wp , take coordinates in $\text{PG}(3, q^4)$ such that Σ (resp. Σ') is the set of all points with coordinates rational over \mathbb{F}_q (resp. \mathbb{F}_{q^2}), and define the image r^\wp of any line r through minors of order two in the usual way. Then $\mathcal{K} = \mathcal{K}' \cap \text{PG}(5, q)$ by considering $\text{PG}(5, q)$ as a subset of $\text{PG}(5, q^2)$. The only nontrivial element of the subgroup of $\text{P}\Gamma\text{L}(6, q^2)$ fixing $\text{PG}(5, q)$ pointwise is

$$\tau : \langle (x_0, x_1, x_2, x_3, x_4, x_5) \rangle_{\mathbb{F}_{q^2}} \mapsto \langle (x_0^q, x_1^q, x_2^q, x_3^q, x_4^q, x_5^q) \rangle_{\mathbb{F}_{q^2}}. \quad (12)$$

Then $\mathcal{K}_2^\tau = \mathcal{K}_2$, and $\sigma\wp = \wp\tau$.

Proposition 3.2. *Let S be a solid in $\text{PG}(5, q^2)$ such that (i) $S \cap \mathcal{K}' \cong Q^-(3, q^2)$, (ii) $S \cap \mathcal{K} = \emptyset$. Then $S \cap S^\tau \cap \mathcal{K}'$ is a set of two distinct points forming an orbit of τ .*

Proof. If $\dim(S \cap S^\tau) \geq 2$, then $S \cap S^\tau$ contains a plane of $\text{PG}(5, q)$. Each plane of $\text{PG}(5, q)$ meets \mathcal{K} in at least one point of $\text{PG}(5, q)$, contradicting (ii). Then $r = S \cap S^\tau$ is a line fixed by τ , so it is a line of $\text{PG}(5, q)$. This r is external to the Klein quadric \mathcal{K} by (ii), hence it meets \mathcal{K}' in two points. Since both of \mathcal{K}' and r are fixed by τ the assertion follows. \square

Proposition 3.3. *There is a line r in $\text{PG}(3, q^4)$, such that r and r^σ are skew lines both meeting ℓ , and $r^{\sigma^2} = r$.*

Proof. Let Σ and Σ' be as in Proposition 3.1. Since $\ell \cap \Sigma' = \emptyset$, ℓ defines a regular (Desarguesian) spread \mathcal{F} of Σ' . The lines of \mathcal{F} are all lines $\langle P, P^{\sigma^2} \rangle \cap \Sigma'$ where $P \in \ell$. The image \mathcal{F}^\wp under the Plücker embedding of \mathcal{F} is an

elliptic quadric $S \cap \mathcal{K}' \cong Q^-(3, q^2)$ in $\text{PG}(5, q^2)$, S a solid. Since \mathbb{L} is scattered, there is no line of \mathcal{F} fixed by σ , whence $S \cap \mathcal{K} = \emptyset$. Then the assertion follows from Proposition 3.2. \square

Theorem 3.4. *Any maximum scattered linear \mathbb{F}_q -linear set in $\text{PG}(1, q^4)$ is projectively equivalent to $L_{U(b)}$ for some $b \in \mathbb{F}_{q^4}$, $N_{q^4/q}(b) \neq 1$.*

Proof. The set $L_{U(0)}$ is a linear set of pseudoregulus type. Now assume that $\mathbb{L} = p_\ell(\Sigma)$ is maximum scattered, not of pseudoregulus type. Coordinates X_0, X_1, X_2, X_3 in $\text{PG}(3, q^4)$ can be chosen such that

$$\Sigma = \{ \langle (u, u^q, u^{q^2}, u^{q^3}) \rangle_{\mathbb{F}_{q^4}} : u \in \mathbb{F}_{q^4}^* \}, \quad (13)$$

and a generator of the subgroup of $\text{PGL}(4, q^4)$ fixing Σ pointwise is

$$\sigma : \langle (x_0, x_1, x_2, x_3) \rangle_{\mathbb{F}_{q^4}} \mapsto \langle (x_3^q, x_0^q, x_1^q, x_2^q) \rangle_{\mathbb{F}_{q^4}}. \quad (14)$$

Define $C = \ell \cap r$, where r is as in Proposition 3.3. The points C and C^{σ^2} lie on r , as well as the points C^σ and C^{σ^3} lie on r^σ . By Proposition 3.1, $C \neq C^{\sigma^2}$ and $C^\sigma \neq C^{\sigma^3}$. This implies $\ell \subset \langle C, C^\sigma, C^{\sigma^3} \rangle$, and $\langle C, C^\sigma, C^{\sigma^2}, C^{\sigma^3} \rangle = \text{PG}(3, q^4)$. Since the stabilizer of Σ in $\text{PGL}(4, q^4)$ acts transitively on the points C of $\text{PG}(3, q^4)$ such that $\langle C, C^\sigma, C^{\sigma^2}, C^{\sigma^3} \rangle = \text{PG}(3, q^4)$ [4, Proposition 3.1], it may be assumed that $C = \langle (0, 0, 1, 0) \rangle_{\mathbb{F}_{q^4}}$, whence

$$\ell = \langle (0, 0, 1, 0), (0, a, 0, -b) \rangle_{\mathbb{F}_{q^4}},$$

for some $a, b \in \mathbb{F}_{q^4}$, not both of them zero. If $a = 0$, then \mathbb{L} is of pseudoregulus type [9, Theorem 2.3], so $a = 1$ may be assumed. For any point $P_u = \langle (u, u^q, u^{q^2}, u^{q^3}) \rangle_{\mathbb{F}_{q^4}}$ in Σ , the plane containing ℓ and P_u has coordinates $[u^{q^3} + bu^q, -bu, 0, -u]$, and this leads to the desired form for the coordinates of \mathbb{L} . \square

4 Orbits

Analogously to the definition of the ΓL -class of linear sets (cf. Definition 2.4 in [7]) we define the GL -class, which will be needed to study $\text{PGL}(2, q^4)$ -equivalence. Note that for any scattered \mathbb{F}_q -linear set the maximum field of linearity is \mathbb{F}_q .

Definition 4.1. *Let L_U be an \mathbb{F}_q -linear set of $\text{PG}(1, q^t)$ of rank t with maximum field of linearity \mathbb{F}_q . We say that L_U is of ΓL -class s [resp.*

GL-class s] if s is the largest integer such that there exist \mathbb{F}_q -subspaces U_1, U_2, \dots, U_s of $\mathbb{F}_{q^t}^2$ with $L_{U_i} = L_U$ for $i \in \{1, 2, \dots, s\}$ and there is no $\varphi \in \Gamma\text{L}(2, q^t)$ [resp. $\varphi \in \text{GL}(2, q^t)$] such that $U_i = U_j^\varphi$ for each $i \neq j$, $i, j \in \{1, 2, \dots, s\}$.

The first part of the following result is [7, Theorem 4.5], while the second part follows from its proof. We briefly summarize the main steps of the proof from [7].

Theorem 4.2. [7, Theorem 4.5] *Each \mathbb{F}_q -linear set of rank four in $\text{PG}(1, q^4)$, with maximum field of linearity \mathbb{F}_q , is of ΓL -class one. More precisely, if $L_U = L_V$ for some 4 dimensional \mathbb{F}_q -subspaces U, V of $\mathbb{F}_{q^4}^2$, then there exists $\varphi \in \Gamma\text{L}(2, q^4)$ such that $U^\varphi = V$. Also, φ can be chosen such that it has companion automorphism either the identity, or $x \mapsto x^{q^2}$.*

Proof. Assume $L_U = L_V$. We may assume $\langle (0, 1) \rangle_{\mathbb{F}_{q^4}} \notin L_U$. Then $U = U_f = \{(x, f(x)) : x \in \mathbb{F}_{q^4}\}$ and $V = V_g = \{(x, g(x)) : x \in \mathbb{F}_{q^4}\}$ for some q -polynomials f and g over \mathbb{F}_{q^4} . By [7, Proposition 4.2], either $g(x) = f(\lambda x)/\lambda$, or $g(x) = \hat{f}(\lambda x)/\lambda$ for some $\lambda \in \mathbb{F}_{q^4}^*$, where here \hat{f} denotes the adjoint map of f with respect to the bilinear form $\langle x, y \rangle := \text{Tr}_{q^4/q}(xy)$. The \mathbb{F}_{q^4} -linear map $\mathbf{v} \mapsto \lambda \mathbf{v}$ maps U_g to one of U_f , or $U_{\hat{f}}$. In the proof of [7, Theorem 4.5], a $\kappa \in \Gamma\text{L}(2, q^4)$ with companion automorphism the identity, or $x \mapsto x^{q^2}$ is determined such that $U_f^\kappa = U_{\hat{f}}$. \square

Theorem 4.3. *For any $b \in \mathbb{F}_{q^4}$, $L_{U(b)}$ is of GL-class one.*

Proof. By Theorem 4.2, if $L_{U(b)} = L_V$, then there exists $\varphi \in \Gamma\text{L}(2, q^4)$ such that $U(b)^\varphi = V$ and the companion automorphism of φ is $x \mapsto x^{q^2}$, or the identity. In order to prove the statement it is enough to show that $U(b)$ and $U(b)^{q^2} = \{(x^{q^2}, y^{q^2}) : (x, y) \in U(b)\}$ lie on the same orbit of $\text{GL}(2, q^4)$. If $b = 0$, then $U(b) = U(b)^{q^2}$. If $b \neq 0$, then for any $u \in \mathbb{F}_{q^4}$,

$$\begin{pmatrix} b^{q^3} & 0 \\ 0 & b^{q^2} \end{pmatrix} \begin{pmatrix} u \\ bu^q + u^{q^3} \end{pmatrix} = \begin{pmatrix} b^q u^{q^2} \\ b(b^q u^{q^2})^q + (b^q u^{q^2})^{q^3} \end{pmatrix}^{q^2} = \begin{pmatrix} v \\ bv^q + v^{q^3} \end{pmatrix}^{q^2},$$

with $v = b^q u^{q^2}$. \square

Corollary 4.4. *Let $b, c \in \mathbb{F}_{q^4}$. The linear sets $L_{U(b)}$ and $L_{U(c)}$ are projectively equivalent if and only if $U(b)$ and $U(c)$ are in the same orbit under the action of $\text{GL}(2, q^4)$.*

Proof. The “if” part is obvious, so assume that $L_{U(b)}^{\tilde{\kappa}} = L_{U(c)}$ where $\kappa \in \text{GL}(2, q^4)$. Then $L_{U(b)\kappa} = L_{U(c)}$ and by Theorem 4.3 there is $\kappa' \in \text{GL}(2, q^4)$ such that $U(b)^{\kappa\kappa'} = U(c)$. \square

It follows that in order to classify the \mathbb{F}_q -linear sets $L_{U(b)}$ up to $\text{PGL}(2, q^4)$ and $\text{P}\Gamma\text{L}(2, q^4)$ -equivalence, it is enough to determine the orbits of the subspaces $U(b)$ under the actions of $\Gamma\text{L}(2, q^4)$ and $\text{GL}(2, q^4)$.

Theorem 4.5. *Let q be a power of a prime p .*

- (i) *For any $b, c \in \mathbb{F}_{q^4}$, $L_{U(b)}$ and $L_{U(c)}$ are equivalent up to an element of $\text{P}\Gamma\text{L}(2, q^4)$ if and only if $c^{q^2+1} = b^{\pm p^s(q^2+1)}$ for some integer $s \geq 0$.*
- (ii) *For any $b, c \in \mathbb{F}_{q^4}$, the linear sets $L_{U(b)}$ and $L_{U(c)}$ are projectively equivalent if and only if $c^{q^2+1} = b^{q^2+1}$ or $c^{q^2+1} = b^{-q(q^2+1)}$.*
- (iii) *All linear sets described in 2. of Theorem 2.1 are projectively equivalent.*
- (iv) *There are precisely $q(q-1)/2$ distinct linear sets up to projective equivalence in the family described in 1. of Theorem 2.1, and these are the only maximum scattered linear sets of $\text{PG}(1, q^4)$.*
- (v) *There are precisely q distinct linear sets up to projective equivalence in the family described in 3. of Theorem 2.1.*

Proof. Take $b \in \mathbb{F}_{q^4}^*$. If $L_{U(b)}$ is not scattered, then it clearly cannot be equivalent to $L_{U(0)}$ (the scattered linear set of pseudoregulus type), while if $L_{U(b)}$ is scattered, then it follows from Theorem 1.1 (and from a computer search when $q = 3$) that $U(b)$ and $U(0)$ yield projectively inequivalent linear sets. Since the automorphic collineations $(x, y) \mapsto (x^{p^s}, y^{p^s})$ fix $U(0)$, it also follows that $L_{U(0)}$ and $L_{U(b)}$ lie on different orbits of $\text{P}\Gamma\text{L}(2, q^4)$. Thus (i) and (ii) are true when one of b or c is zero, so from now on we may assume $b \neq 0$ and $c \neq 0$.

The sets $L_{U(b)}$ and $L_{U(c)}$ are equivalent up to elements of $\text{P}\Gamma\text{L}(2, q^4)$ if and only for some $\psi = p^k$, $k \in \mathbb{N}$ and some $A, B, C, D \in \mathbb{F}_{q^4}$ such that $AD - BC \neq 0$ the following holds:

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u^\psi \\ b^\psi u^{\psi q} + u^{\psi q^3} \end{pmatrix} : u \in \mathbb{F}_{q^4} \right\} = \left\{ \begin{pmatrix} v \\ cv^q + v^{q^3} \end{pmatrix} : v \in \mathbb{F}_{q^4} \right\}. \quad (15)$$

Furthermore, by Corollary 4.4, $L_{U(b)}$ and $L_{U(c)}$ are projectively equivalent if, and only if, (15) has a solution with $\psi = 1$. This leads to a polynomial in

u^ψ of degree at most q^3 which is identically zero. Equating its coefficients to zero,

$$\begin{cases} A^{q^3} - D &= 0 \\ B^q b^{\psi q} c + B^{q^3} &= 0 \\ A^q c - D b^\psi &= 0 \\ B^q c + B^{q^3} b^{\psi q^3} - C &= 0. \end{cases} \quad (16)$$

Assume that $L_{U(b)}$ and $L_{U(c)}$ are in the same orbit of $\text{PGL}(2, q^4)$, and take $\psi = 1$ in case they are also projectively equivalent. If $D \neq 0$, then the first and third equations imply $b^\psi = D^{q^2-1}c$ and so $c^{q^2+1} = b^{\psi(q^2+1)}$. If $D = 0$, then $BC \neq 0$; from the second equation, $(b^{\psi q}c)^{q^2+1} = 1$, hence $c^{q^2+1} = b^{-\psi q(q^2+1)}$. This proves the only if parts of (i) and (ii).

Conversely, if $c^{q^2+1} = b^{p^s(q^2+1)}$ for some $s \in \mathbb{N}$, then $b^{p^s}c^{-1} = \delta^{q^2-1}$ for some $\delta \in \mathbb{F}_{q^4}^*$. The quadruple $A = \delta^q$, $B = C = 0$, $D = \delta$ with $\psi = p^s$ is a solution of (16) with $AD - BC \neq 0$. This proves the if part of (i) when $c^{q^2+1} = b^{p^s(q^2+1)}$ and the if part of (ii) when $c^{q^2+1} = b^{q^2+1}$. If $b^{q^2+1} = c^{q^2+1} = 1$, i.e. when $U(b)$ and $U(c)$ define linear sets described in 2. of Theorem 2.1, then the above condition holds, thus (iii) follows. From now on we may assume $b^{q^2+1} \neq 1$ and $c^{q^2+1} \neq 1$.

Assume $c^{q^2+1} = b^{-p^s(q^2+1)}$ for some $s \in \mathbb{N}$, i.e. $b^{p^s}c = \varepsilon^{q^2-1}$ for some $\varepsilon \in \mathbb{F}_{q^4}^*$. Define $\psi = p^s q^3$. A $\rho \in \mathbb{F}_{q^4}^*$ exists such that $\rho^{q^2-1} = -1$. Take $A = D = 0$, $B = (\rho\varepsilon)^{q^3}$, $C = \varepsilon\rho c(1 - b^{p^s(q^2+1)})$. If $C = 0$, then $b^{q^2+1} = 1$, a contradiction. So $AD - BC \neq 0$ and (16) has a solution. If $p^s = q$, then $\psi = 1$, hence in this case $L_{U(b)}$ and $L_{U(c)}$ are projectively equivalent. This finishes the proofs of (i) and (ii).

Now we prove (iv). Note that $N_{q^4/q}(b) = (b^{q^2+1})^{q+1}$ for any $b \in \mathbb{F}_q$, therefore, $L_{U(b)}$ is a maximum scattered \mathbb{F}_q -linear set not of pseudoregulus type if, and only if, b^{q^2+1} is an element of the set

$$S = \{x \in \mathbb{F}_{q^2}^* : x^{q+1} \neq 1\}.$$

The orbits of point sets of type $L_{U(b)}$, $b \neq 0$, under the action of $\text{PGL}(2, q^4)$ are as many as the pairs $\{x, x^{-q}\}$ of elements in S . Since all such pairs are made of distinct elements, adding one for the linear set of pseudoregulus type, one obtains

$$1 + \frac{q^2 - q - 2}{2} = \frac{q(q-1)}{2}.$$

Finally we prove (v). $L_{U(b)}$ is an \mathbb{F}_q -linear set described in 3. of Theorem 2.1 if, and only if, b^{q^2+1} is an element of the set

$$Z = \{x \in \mathbb{F}_{q^2} \setminus \{1\} : x^{q+1} = 1\}.$$

The orbits of point sets of this type under the action of $\text{PGL}(2, q^4)$ are as many as the pairs $\{x, x^{-q}\}$ of elements in Z . Since for each $x \in Z$ we have $x = x^{-q}$, this number is q . \square

Remark 4.6. *The number of orbits of maximum scattered linear sets under the action of $\text{PGL}(2, q^4)$ depends on the exponent e in $q = p^e$. A general formula is not provided here. For $e = 1$ each orbit which does not arise from the linear set of pseudoregulus type is related to two or four norms over \mathbb{F}_{q^2} , according to whether $N_{q^4/q^2}(b) \in \mathbb{F}_q \setminus \{0, 1, -1\}$ or not. This leads (including now the linear set of pseudoregulus type) to a total number of $(q^2 - 1)/4$ orbits for odd q .*

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